On Weierstrass semigroups arising from finite graphs

Justin D. Peachey*

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Abstract

In 2007, Baker and Norine proved an analogue of the Riemann-Roch Theorem for finite graphs. Motivated by the Weierstrass semigroup of a point on a nonsingular projective curve X, it is natural to consider analogues for a vertex P on a finite, connected graph which has no loops. Let $H_r(P)$ be the set of nonnegative integers α such that $r(\alpha P) = r((\alpha-1)P) + 1$, where$r(D)$ denotes the dimension of a divisor D, and let $H_f(P)$ be the set of nonnegative integers α such that there exists an integer-valued function f on the vertices of a graph so that f has a pole only at P of order α. If P is a point on a curve X and $r(\alpha P)$ is taken to be the dimension of the divisor αP, then these two sets are equal; indeed, $H_r(P)$ is well-studied Weierstass semigroup of P. However, in the case where P is a vertex of a finite graph G, these two sets may not be equal. In this paper, we explore the relationship between these sets.

1 Introduction

The Weierstrass semigroup of a place on a function field is an object of classical study and is tied to the dimension of associated Riemann-Roch spaces. It is of particular interest in coding theory as elements of the Weierstrass semigroup provide codes with better parameters. In this paper, we consider analogues of Weierstrass semigroups arising from finite graphs and study the relationship between these analogues.

Let G be a finite connected graph containing no loops. If D is a divisor on G, let the dimension of D, $r(D)$, be given by

$$r(D) = \max\left\{k : \text{ there exists } f \in \mathcal{M}(G) \text{ such that } \Delta(f) \geq E - D \text{ for all } E \in Div_+^k(G)\right\},$$

where $\mathcal{M}(G)$ denotes the set of integer valued functions on $V(G)$ and $\Delta(f)$ represents the divisor of f. Baker and Norine [3] proved an analogue of the Riemann-Roch Theorem for finite graphs: $r(D) = \deg(D) + 1 - g + r(K - D)$ where K is a canonical divisor and g is the genus of the graph.

In [2], Baker defined a Weierstrass gap to be $\alpha \in \mathbb{N}$ such that $r(\alpha P) = r((\alpha-1)P)$. Applying this definition, we consider a set

$$H_r(P) = \{\alpha \in \mathbb{N} \ : \ r(\alpha P) = r((\alpha-1)P) + 1\},$$

which represents the complement of the set of gaps.

We also consider a second generalization of Weierstrass gaps arising from integer-valued functions on the vertices of a graph. The resulting complement is a numerical semigroup defined as

$$H_f(P) = \left\{\alpha \in \mathbb{N} : \begin{array}{c} \text{there exists } f \in \mathcal{M}(G) \text{ such that} \\ \Delta(f) = A - \alpha P \text{ where } A \geq 0 \text{ and } P \notin \text{supp} A \end{array}\right\}.$$

Furthermore, we show how this second definition can be used to study the one arising from [2]. This paper is organized as follows. This section concludes with a brief note on notation. Section 2 reviews information

*Department of Mathematics, Davidson College, Davidson, NC 28035 email: jupeachey@davidson.edu

on Weierstrass semigroups and function fields and includes a overview of the Riemann-Roch theory on a finite graph. Sections 3 and 4 contain the main results of this paper.

Notation. The set of nonnegative integers is denoted by $\mathbb{N}$, and $\mathbb{Z}^+$ denotes the set of positive integers. Furthermore, given $a_1, \ldots, a_k \in \mathbb{Z}^+$, the (numerical) semigroup generated by $a_1, \ldots, a_k$ is

$$\langle a_1, \ldots, a_k \rangle := \left\{ \sum_{i=1}^{k} c_i a_i : c_i \in \mathbb{N} \right\}.$$

We say that α is a gap of $\langle a_1, \ldots, a_k \rangle$ if and only if $\alpha \in \mathbb{N} \setminus \langle a_1, \ldots, a_k \rangle$. All graphs in this paper are finite, connected, and without loops. Hence, we say a graph to mean a graph satisfying these conditions. Given a graph G, $V(G)$ is the vertex set of G, and $E(G)$ denotes the edge set of G. Given a matrix A and $i, j \in Z^+$, $A_i j$ denotes the entry of A in the i^{th} row and j^{th} column.

2 Background

2.1 Function Fields and Weierstrass semigroups

In this section, we review the necessary information on Weierstrass semigroups in the algebraic geometric setting. For more information, see [1], [5], [7].

Let $\mathbb{F}$ be a finite field and $F/\mathbb{F}$ be an algebraic function field of genus $g > 1$. A divisor of $F/\mathbb{F}$ is an element of the free abelian group which is generated by the places of $F/\mathbb{F}$. Let $f \in F \setminus \{0\}$ and denote by Z (respectively N) the set of zeros (poles) of f in $\mathbb{P}_F$. Then, the divisor of a function $f \in F \setminus \{0\}$ is

$$(f) = \sum_{P \in Z} v_P(f) P + \sum_{P \in N} v_P(f) P$$

(resp. $(f)_\infty = \sum_{P \in N} (-v_P(f))P)$ where v_P denotes the discrete valuation corresponding to P. The Riemann-Roch space of a divisor A of F is

$$\mathcal{L}(A) := \{f \in F \setminus \{0\} : (f) \geq -A\} \cup \{0\}.$$

The Riemann-Roch space $\mathcal{L}(A)$ is a finite-dimensional vector space over $\mathbb{F}$; let $\ell(A)$ denote the dimension of the vector space $\mathcal{L}(A)$ over $\mathbb{F}_q$. Then Riemann-Roch Theorem states that

$$\ell(A) = \deg A + 1 - g + \ell(W - A)$$

where W is any canonical divisor of F. Moreover, if the degree of A is at least $2g - 1$, then $\ell(W - A) = 0$ and so $\ell(A) = \deg A + 1 - g$. In [1], the authors introduced the idea of a Weierstrass semigroup of multiple points.

Definition 2.1. *Given an algebraic function field $F/\mathbb{F}$ and distinct places $P_1, \ldots, P_m$ of F of degree one, the Weierstrass semigroup of the m-tuple $(P_1, \ldots, P_m)$ is*

$$H(P_1, \ldots, P_m) = \left\{ (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^m : \begin{array}{l} \textit{there exists } f \in F \textit{ with } (f) = A - \sum_{i=1}^{r} \alpha_i P_i, \\ \textit{where } A \geq 0 \textit{ and } P_i \notin \textit{supp}A, 1 \leq i \leq r \end{array} \right\}$$

If $m = 1$, we obtain the classical Weierstrass semigroup

$$H(P) := \left\{ \alpha \in \mathbb{N} : \begin{array}{l} \textit{there exists } f \in F \textit{ with } (f) = A - \alpha P \\ \textit{where } A \geq 0 \textit{ and } P \notin \textit{supp}A \end{array} \right\}.$$

The following proposition provides a relationship between the dimension of the Riemann-Roch space of αP and $\alpha \in H(P)$.

Proposition 2.2. *Given a rational place P,*

$$H(P) = \{\alpha \in \mathbb{N} : \ell(\alpha P) = \ell((\alpha - 1)P) + 1\}. \tag{1}$$

Thus, by the Riemann-Roch theorem, if $\alpha \geq 2g$, $\alpha \in H(P)$. Thus, $G(P)$ is finite; in fact, $G(P) \subseteq [0, 2g-1]$ and $|G(P)| = g$.

We may generalize (1) to multiple places as the following result shows.

Lemma 2.3. *[7, Lemma 8] Let $\alpha \in \mathbb{N}^m$ and let $P_1, ..., P_m$ be distinct rational places of F/K. Then, the following are equivalent:*

1. *$\alpha \in H(P_1, \ldots, P_m)$.*
2. *$\ell\left(\sum_{i=1}^m \alpha_i P_i\right) \neq \ell\left((\alpha_j - 1) P_j + \sum_{i=1, i\neq j}^m \alpha_i P_i\right)$, for all $1 \leq j \leq m$.*

We will later seek to extend these results to finite graphs as well.

2.2 Riemann-Roch on finite graphs

In this section, we follow the construction given by Baker and Norine in [3]. Let $V(G) = \{P_1, P_2, \ldots, P_n\}$ be the set of vertices of a graph G. Let $Div(G)$ be the free abelian group on $\{P_1, P_2, \ldots, P_n\}$, the set of vertices of G. Then, a divisor D is an element of $Div(G)$, that is,

$$D = \sum_{i=1}^{n} a_i P_i,$$

where $a_i \in \mathbb{Z}$. The divisor D is effective if and only if $a_i \geq 0$ for all $P_i \in V(G)$, and the degree of D is defined to be $\deg(D) = \sum_{i=1}^{n} a_i$. Furthermore, the support of the divisor D is defined by *supp* $D := \{P \in V(G) : a_i \neq 0\}$. Then, we denote the subset of all effective divisors of degree k by $Div_+^k(G) = \{D \in Div(G) : D \geq 0 \text{ and } \deg(D) = k\}$.

Consider the abelian group of integer-valued functions on the vertices of G,

$$\mathcal{M}(G) = Hom\{V(G), \mathbb{Z}\}.$$

As in the function field setting, we would like to consider the divisor of $f \in \mathcal{M}(G)$. Let $Nbhd(v)$ denote the set of all vertices adjacent to v. If $f \in \mathcal{M}(G)$, the divisor $\Delta(f)$ is

$$\Delta(f) = \sum_{v \in V(G)} \Delta_v(f) v$$

where

$$\Delta_v(f) = \sum_{w \in Nbhd(v)} (f(v) - f(w)) \in \mathbb{Z}.$$

Let A be the adjacency matrix of the graph G, that is, $A \in \mathbb{Z}^{n\times n}$ where A_{ij} is the number of edges between P_i and P_j. Let D is the matrix defined so that $D_{ij} = 0$ if $i \neq j$ and $D_{ii} = \deg(P_i)$. Then, the Laplacian of G is the matrix $Q = D - A$. Hence,

$$Q_{ij} = \begin{cases} -|\{\text{edges between } P_i \text{ and } P_j\}| & \text{if } i \neq j \\ \deg(P_i) & \text{if } i = j \end{cases}.$$

Let $(P_1, P_2, \ldots, P_n)$ be the ordered set of vertices of a graph G. Then, if $f \in \mathcal{M}(G)$, we define $[f] \in \mathbb{Z}^n$ by $[f]_i = f(P_i)$. Similarly, if $D \in Div(G)$ where $D = \sum_{i=1}^n a_i P_i$, $[D]_i = a_i$. If Q is the Laplacian of G,

$$[\Delta(f)] = Q\,[f].$$

Example 2.4. *Consider the graph G given in Figure 2.1. Then*

$$\mathcal{M}(G) = \{f : f(P_i) \in \mathbb{Z} \text{ for } i = 1, 2, 3, 4\}.$$

Consider $f \in \mathcal{M}(G)$ such that $f(P_1) = -1$, $f(P_3) = -1$, $f(P_2) = 0$, and $f(P_4) = 0$.

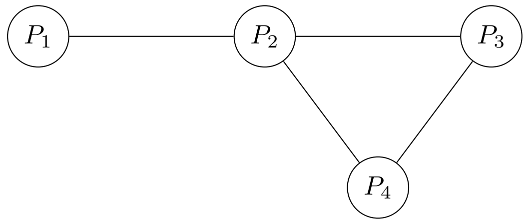

Figure 2.1: Example 2.4

Then, $\Delta_{P_1}(f) = -1 - 0 = -1, \Delta_{P_2}(f) = (0-(-1)) + (0-(-1)) = 2, \Delta_{P_3}(f) = (-1-0) + (-1-0) = -2,$ *and* $\Delta_{P_4}(f) = 0 - (-1) = 1$. *Thus,* $\Delta(f) = 2P_2 + P_4 - P_1 - 2P_3$.

Thus we may view $\Delta(f)$ as an analogue of (f) and $\Delta_v(f)$ as an analogue of $v_P(f)$ in the function field case. We have the following result which follows from the rank of the Laplacian.

Proposition 2.5. *Let G be a graph and $f \in \mathcal{M}(G)$. Then,*

$$\deg(\Delta(f)) = 0.$$

Proof. Note that $\deg(\Delta(f)) = \sum_v \Delta_v(f) = \sum_v \sum_{w \in Nbhd(v)} (f(v) - f(w))$. Thus, it must be that $\deg(\Delta(f)) = \frac{1}{2} \sum_{\substack{v \in V(G) \\ w \in Nbhd(v)}} [(f(v) - f(w)) + (f(w) - f(v))] = 0$. □

Now, we consider the analogue of the dimension of a divisor on a function field.

Definition 2.6. *The dimension of a divisor $D \in Div(G)$ is*

$$r(D) = \max\left\{k : \text{ there exists } f \in \mathcal{M}(G) \text{ such that } \Delta(f) \geq E - D \text{ for all } E \in Div_+^k(G)\right\}.$$

As in the algebraic geometric setting, there exists a relationship between the dimension of a divisor and its degree. This is known as the Riemann-Roch Theorem for finite graphs and was shown in [3]. One may note the similarity between this and (1).

Theorem 2.7. *[3, Theorem 1.12] Let G be a graph, $D \in Div(G)$, $K = \sum_{v \in V(G)} (\deg(v) - 2)v$, and $g = |E(G)| - |V(G)| + 1$. Then,*

$$r(D) = \deg(D) + 1 - g + r(K - D).$$

3 Weierstrass semigroups of a single vertex on finite graphs

In this section we focus on two analogues of $H(P)$ in the case of a finite graph. Both analogues satisfy one of the properties of the Weierstrass semigroup. We are particularly interested in the relationship between these sets and families of graphs for which they coincide.

Definition 3.1. *Let G be a graph and $P \in V(G)$. Then,*

$$H_r(P) = \{\alpha \in \mathbb{N} \ : \ r(\alpha P) = r((\alpha - 1)P) + 1\}$$

and

$$H_f(P) = \left\{\alpha \in \mathbb{N} : \begin{array}{c} \text{there exists } f \in \mathcal{M}(G) \text{ such that} \\ \Delta(f) = A - \alpha P \text{ where } A \geq 0 \text{ and } P \notin supp A \end{array}\right\}.$$

In [4] and [6], the authors consider the image of the Laplacian over $\mathbb{Z}$ for several families of graphs. However, less is known about $H_r(P)$. We study the relationship between $H_f(P)$ and $H_r(P)$ in hopes of uncovering the structure of $H_r(P)$. Recall that given a place P of a function field $F/\mathbb{F}$, $\alpha \in G(P)$ if and only if $\ell(\alpha P) = \ell((\alpha-1)P)$ if and only if there does not exist $f \in F$ such that $(f)_\infty = \alpha P$. We are interested in the analogous relationships (or lack thereof) in the case of a finite graph. Thus, we let

$$G_f(P) = \mathbb{N} \setminus H_f(P)$$

and

$$G_r(P) = \mathbb{N} \setminus H_r(P).$$

Now, applying Theorem 2.7, we can obtain a result similar to the classical Weierstrass gap theorem.

Lemma 3.2. *[2, Lemma 4.2] Let G be a graph, and let $P \in G$. If $\alpha \geq 2g$, $\alpha \in H_r(P)$. Hence, $G_r(P)$ is finite. In fact,*

$$|G_r(P)| = g$$

and $G_r(P) \subseteq [0, 2g-1]$.

The following result examines the relationship between $H_r(P)$ and $H_f(P)$.

Theorem 3.3. *Let G be a graph and P be a vertex of G. Then,*

$$H_r(P) \subseteq H_f(P).$$

Thus, $G_f(P) \subseteq G_r(P)$ and $|G_f(P)| \leq g$.

Proof. Let $\alpha \in H_r(P)$. Then, $r((\alpha-1)P) = k$ and $r(\alpha P) = k+1$. Thus, since $r((\alpha-1)P) = k$, there exists $E_0 \in Div_+^{k+1}(G)$ so that for all $f \in \mathcal{M}(G)$, $(\alpha-1)P - E_0 + \Delta(f) \ngeq 0$.

By definition, for all $E \in Div_+^{k+1}(G)$, there exists $f \in \mathcal{M}(G)$ so that $\alpha P - E + \Delta(f) \geq 0$. Thus, there exists $h \in \mathcal{M}(G)$ so that $\alpha P - E_0 + \Delta(h) \geq 0$. Thus, it must be that $\Delta_P(h) = \alpha$ and $\Delta_v(h) \geq 0$ for all $v \in V(G) \setminus \{P\}$. Hence, $\alpha \in H_f(P)$. ☐

Since $H(P)$ is a semigroup in the function field case, it is natural to as if $H_r(P)$ and $H_f(P)$ are semigroups. We do have the following which follows directly from the Laplacian representation of $\Delta(f)$.

Proposition 3.4. *Let G be a graph and P be a vertex of G. If $\alpha, \beta \in H_f(P)$, $\alpha + \beta \in H_f(P)$.*

Proof. Since $\alpha, \beta \in H_f(P)$, there exist f_1, f_2 such that $\Delta(f_1) = A_1 - \alpha P$ and $\Delta(f_2) = A_2 - \beta P$ where $A_1, A_2 \geq 0$. Thus,

$$\begin{aligned} \Delta_v(f_1+f_2) &= \sum_{w \in Nbhd(v)} ((f_1+f_2)(v) - (f_1+f_2)(w)) \\ &= \sum_{w \in Nbhd(v)} (f_1(v) - f_1(w)) + \sum_{w \in Nbhd(v)} (f_2(v) - f_2(w)) \end{aligned}$$

Thus, we may conclude that

$$\Delta(f_1+f_2) = \Delta(f_1) + \Delta(f_2),$$

that is, $\alpha + \beta \in H_f(P)$. ☐

It remains to determine the algebraic structure of $H_r(P)$. In order to address this for certain families of graphs, we will make extensive use of a certain set of functions in $\mathcal{M}(G)$. The divisors of these functions correspond to the generators Δ_i given in [4].

Definition 3.5. *Let G be a graph with vertex set $V(G) = \{P_1, P_2, \ldots, P_n\}$. The indicator function f_{P_i} is defined by*

$$f_{P_i}(P_j) = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Consider the set

$$\mathcal{B} = \{f_{P_1}, f_{P_2}, \ldots, f_{P_n}\}.$$

Then,

$$\Delta(f_{P_i}) = \sum_{j=1}^{\deg(P_i)} Q_j - \deg(P_i)P_i,$$

where the vertices Q_j are the distinct neighbors of P_i.

These indicator functions represent a single vertex firing in a chip-firing game. When we consider $\Delta_P(f)$, we see that each neighbor of P is gaining a single "zero" and P has a "pole" of order $\deg(P)$. Furthermore, we have the following useful result, which follows from the Laplacian representation of $\Delta(f)$.

Lemma 3.6. *Consider a graph G. Then, if $h \in \mathcal{M}(G)$, $h = \sum_{\mathcal{B}} \alpha_{P_i} f_{P_i}$, where $\alpha_i \in \mathbb{Z}$ for all i.*

Using the set of indicator functions $\mathcal{B}$, we obtain the following result.

Proposition 3.7. *Let G be a graph and P be a vertex of G such that the subgraph G' induced by $V(G) \setminus \{P\}$ is connected. Then, $\deg(P)$ is the smallest nonzero element of $H_f(P)$.*

Proof. Consider f_P. Then, $\Delta(f_P) = A - \deg(P)P$, where $A \geq 0$. Thus, $\deg(P) \in H_f(P)$. It remains to show it is the minimum nonzero element of $H_f(P)$.

Suppose $c \in H_f(P)$ and $c < \deg(P) = d$. We may assume $h \in M(G)$ such that $\Delta(h) = A - cP$, where $A \geq 0$.

If $k \in \mathbb{Z}$, $\Delta(h) = \Delta(h + k)$ as

$$\Delta_P(h+k) = \sum_{w \in Nbhd(P)} ((h(P) + k) - (h(w) + k)) = \sum_{w \in Nbhd(P)} (h(P) - h(w)) = \Delta_P(h).$$

Consider $v \in V(G)$ so that

$$h(v) = \max\{h(w) | w \in V(G)\}.$$

Then, $(h - h(v))(P_i) \leq 0$ for $P_i \in V(G)$ and $\Delta(h) = \Delta(h - h(v))$. Thus, $\Delta_P(h) = \sum_w \alpha_w - \alpha_P \deg(P)$, where w is a neighbor of P. Hence, we may assume $h(w) \leq 0$ for all $w \in V(G)$.

Let $v \in V(G)$ be chosen so that $\alpha_v = \min\{\alpha_w | w \in V(G) \setminus \{P\}\}$. Since f has no poles at $v \in V(G')$, we must have $\alpha_v = \alpha_{v_i}$ for all neighbors v_i of v. Repeating this argument and using that G' is connected we must have that $\alpha_w = \alpha_v$ for all $w \in V(G')$. Thus, $c = \sum_w \alpha_v - \alpha_P \deg(P) = \deg(P)(\alpha_P - \alpha_v)$. Hence, $\deg(P) | c$, which means $c = 0$. Thus, $\deg(P)$ is the smallest nonzero element of $H_f(P)$. □

This result also demonstrates that if $c \in H_r(P)$, $c \geq \deg(P)$. Furthermore, applying the previous proposition to C_n gives us the following result.

Proposition 3.8. *Consider the cycle on n vertices, C_n. Then,*

$$H_r(P) = H_f(P) = \langle 2, 3 \rangle$$

for all $P \in V(C_n)$.

Proof. Consider $P \in V(C_n)$. By Theorem 2.7 and the fact that $g = 1$, we know that if $\alpha \geq 2$, $\alpha \in H_r(P)$. Thus, $H_r(P) = \langle 2, 3 \rangle$.

Now, since $H_r(P) \subseteq H_f(P)$, $\langle 2, 3 \rangle \subseteq H_f(P)$. It remains to show that $1 \notin H_f(P)$.

We know that $\deg(P) = 2$ is the minimum element of $H_f(P) \setminus \{0\}$ by Proposition 3.7. Thus, $1 \notin H_f(P)$ and $H_f(P) = \langle 2, 3 \rangle$. □

Since C_n is regular, this motivates us to consider several other families of regular graphs. The next candidate is K_n, the complete graph on n vertices. First, we show the following lemma which will be of use later.

Lemma 3.9. *Consider the complete graph, K_n. Then, $\langle n-1, n\rangle \subseteq H_f(P)$ for all $P \in V(K_n)$.*

Proof. By Proposition 3.7, $n-1 \in H_f(P)$ and $\Delta(f_P) = \sum_{i=1}^{n-1} P_i - (n-1)P$.

Now, consider $f = f_P - f_{P_2}$ where $P, P_2 \in V(G)$. Then,

$$\Delta_P(f) = -1\deg(P) - 1$$

and

$$\Delta_w(f) \geq 0$$

for all $w \in V(G) \setminus \{P_1\}$. Thus, $n \in H_f(P)$. The desired result then follows by Proposition 3.4. □

If $\alpha \notin \langle n-1, n\rangle$, then $\alpha = i(n-1)+j$ where $0 \leq i < j \leq n-2$. Furthermore, α must be element of the following array.

$$\begin{array}{ccccccc}
1 & 2 & 3 & \cdots & n-4 & n-3 & n-2 \\
n+1 & n+2 & n+3 & \cdots & 2n-4 & 2n-3 & \\
2n+1 & 2n+2 & 2n+3 & \cdots & 3n-4 & & \\
\vdots & & & \iddots & & & \\
\vdots & & \iddots & & & & \\
\vdots & \iddots & & & & & \\
(n-1)n+1 & & & & & &
\end{array}$$

Thus, $\alpha = kn+l$ where $1 \leq l \leq n-2-k$ and $0 \leq k \leq (n-1)$. Rewriting this we get that $k(n-1)+l+k$. Hence, if $i = k$ and $j = l+k$, we get that $0 \leq i < j \leq n-2$. We may conclude that if $\alpha \notin \langle n-1, n\rangle$, then $\alpha = i(n-1)+j$ where $0 \leq i < j \leq n-2$. Counting these elements we get there are $\frac{(n-1)(n-2)}{2}$ gaps of $\langle n-1, n\rangle$. We now have enough information to show the desired result for K_n.

Proposition 3.10. *If $G = K_n$ and $P \in V(G)$,*

$$H_r(P) = H_f(P) = \langle n-1, n\rangle.$$

Proof. We show that $H_f(P) = \langle n-1, n\rangle$ and the proposition follows. By Lemma 3.9, $\langle n-1, n\rangle \subseteq H_f(P)$.

First, we have $g = |E(G)| - |V(G)| + 1$. Since $|V(G)| = n$ and $|E(G)| = \frac{n(n-1)}{2}$, $g = \frac{n(n-1)}{2} - n + 1$. Thus, we have the following:

$$g = \binom{n}{2} - n + 1 = \frac{(n-1)(n-2)}{2}.$$

Suppose $\alpha \in H_f(P)$ and $\alpha \notin \langle n-1, n\rangle$. Then, there exists an $h \in \mathcal{M}(G)$ such that $\Delta(h) = A - \alpha P$ where $A \geq 0$. Then, $\alpha = i\deg(P) + j$ where $0 \leq i < j \leq n-2$. As before, we may assume that $h(v) \leq 0$ for all $w \in V(G)$ or equivalently that $\alpha_w \geq 0$ for all $v \in V(G)$.

Now, using the construction of h given in Lemma 3.6, we get that

$$\Delta(h) = \sum_v \left(\sum_{w \neq v} \alpha_w - \alpha_v \deg(v) \right) v.$$

By assumption, we have

$$\sum_{w \neq P} \alpha_w - \alpha_P \deg(P) = -i\deg(P) - j$$

where $0 \leq i < j \leq \deg(P) - 1$. Hence, $\sum\limits_{w \neq P} \alpha_w \equiv -j \mod \deg(P)$.

Since $h(w) \leq 0$ for $w \in V(G)$, $\alpha_v \geq 0$ for all $v \in V(G)$, i.e., $\sum\limits_{w \neq P} \alpha_w \neq -j$. Thus, it must be that $\sum\limits_{w \neq P} \alpha_w = l(n-1) - j$ where $l > 0$.

Now, since $l \geq 1$ and $\sum_{w \neq P} \alpha_w - \alpha_P \deg(P) = -i \deg(P) - j$, $\alpha_p = i + l$. Choose v so that $\alpha_v = \max\{\alpha_w | w \in V(G) \setminus \{P\}\}$. Recall that $\alpha_v > 0$ by assumption. Consider $\Delta_v(h)$. Then,

$$\begin{aligned} \Delta_v(h) &= \sum_{w \neq v, P} \alpha_w + \alpha_P - \alpha_v(n-1) \\ &= \sum_{w \neq P} \alpha_w - \alpha_v + \alpha_P - \alpha_v(n-1) \\ &= -j + l(n-1) + i + l - \alpha_v(n) \\ &= i - j + l(n) - \alpha_v(n) \\ &= i - j + (l - a_v)(n) \end{aligned}$$

Suppose $\alpha_v \leq l - 1$. Thus,

$$\sum_{w \neq P} \alpha_w \leq \sum_{i=1}^{deg(P)} a_v \leq \sum_{i=1}^{deg(P)} (l-1) = (l-1)\deg(P) < -j + l\deg(P),$$

which is a contradiction. Hence, $\alpha_v \geq l$. Since $i - j < 0$ and $l - a_v \leq 0$, $\Delta_v(h) < 0$ which contradicts that $\Delta(h) = A - \alpha P$ where $A \geq 0$. Thus, it must be that no such α exists. Therefore, $H_f(P) = \langle n-1, n \rangle$; thus, $H_f(P)$ is a numerical semigroup with g gaps. Since $H_r(P) \subseteq H_f(P)$ and $H_r(P)$ also has g gaps by Proposition 3.2, $H_r(P) = H_f(P) = \langle n-1, n \rangle$. $\square$

Lemma 3.11. *Let (U, V) be the natural partition of the vertices of the complete bipartite graph $K_{m,n}$ where $|U| = m$ and $|V| = n$. Then, if $P \in U$,*

$$\langle n, (m-1)n + 1, (m-1)n + 2, \ldots, (m-1)n + (n-1) \rangle \subseteq H_f(P).$$

Proof. We show this for $P \in U$ as the argument is similar for $P \in V$. Let $V = \{Q_1, \ldots, Q_n\}$ and $U = \{P, P_1, \ldots, P_{m-1}\}$.

Consider the set of indicator functions B defined on G; that is, $f_v(v) = -1$ and $f_v(w) = 0$ for all $w \in V(G) \setminus \{v\}$. Then,

$$\Delta_{Q_i}(f_P) = 1$$

for $Q \in V$,

$$\Delta_{P_i}(f_P) = 0$$

for $P_i \in U \setminus \{P\}$, and

$$\Delta_P(f_P) = -n.$$

Thus, $n \in H_f(P)$. as $\Delta(f_P) = \sum_V Q_i - nP$. Next, we consider $h = mf_P + \sum\limits_{i=1}^{l} f_{Q_i}$, where $1 \leq l \leq n-1$. Thus, $1 \leq n - l \leq n - 1$. Note the following:

$$\begin{aligned} \Delta_{Q_j}(h) &= m\Delta_{Q_j}(f_P) + \sum_{i=1}^{l} \Delta_{Q_j}(f_{Q_i}) = 0 \text{ where } 1 \leq j \leq l \\ \Delta_{Q_j}(h) &= m - 0 \text{ where } j > l \\ \Delta_{P_i}(h) &= l, \text{ and} \\ \Delta_P(h) &= m\Delta_{Q_j}(f_P) + \sum_{i=1}^{l} \Delta_{Q_j} f_{Q_i} = -((m-1)n + n - l). \end{aligned}$$

Hence, since $\Delta_v(h) \geq 0$ for $v \in V(G) \setminus \{P\}$, we obtain the desired result by applying Proposition 3.4. $\square$

Now, consider $a \in \mathbb{Z} \cap [1, (m-1)n]$ and let

$$N = \langle n, (m-1)n+1, (m-1)n+2, \ldots, (m-1)n+(n-1) \rangle.$$

As $a \in [1, (m-1)n]$, $n|a$ if $a \in N$. Thus, the gaps of N are

$$\{in + j | 0 \leq i \leq m-2,\ 1 \leq j \leq n-1\}.$$

We can apply our results to $K_{m,n}$ to obtain the following result.

Proposition 3.12. *Let (U, V) be the natural partition of the vertices of the complete bipartite graph $K_{m,n}$ where $|U| = m$ and $|V| = n$. Then, if $P \in U$,*

$$H_f(P) = H_r(P) = \langle n, (m-1)n+1, (m-1)n+2, \ldots, (m-1)n+(n-1) \rangle.$$

Proof. Let $N = \langle n, (m-1)n+1, (m-1)n+2, \ldots, (m-1)n+(n-1) \rangle$. By Lemma 3.11, $N \subseteq H_f(P)$. Furthermore, the genus of $K_{m,n}$ is $g = |E(G)| - |V(G)| + 1$. Thus, $g = mn - (m+n) + 1 = (m-1)(n-1)$, and

$$|\mathbb{N} \setminus N| = (m-1)n - (m-1) = (m-1)(n-1) = g$$

by Lemma 3.11.

Thus, if we can show that $N = H_f(P)$, the theorem is proved. Suppose that

$$N \subsetneqq H_f(P),$$

that is, there exists $\alpha \in H_f(P)$ but $\alpha \notin N$. Hence, there exists $h \in M(G)$ so that $\Delta(h) = A - \alpha P$, $P \notin supp\ A$, $A \geq 0$. Note, as above, we may assume that $h(w) \leq 0$ for all $w \in V(G)$ or equivalently, $\alpha_w \geq 0$ for all $w \in V(G)$.

As before, we have $\Delta(h) = \sum_v \left(\sum_{w \in Nbhd(v)} \alpha_w - \alpha_v \deg(v) \right) v$. Furthermore, we know $\alpha = in + j$ where $0 \leq i \leq m-2,\ 1 \leq j \leq n-1$. Combining these facts,

$$\sum_{i=1}^{n} \alpha_{Q_i} - \alpha_P n = -in - j.$$

Hence,

$$\sum_{i=1}^{n} \alpha_{Q_i} \equiv -j \mod n.$$

Since $\alpha_{Q_i} \geq 0$ for all i, $\sum_{i=1}^{n} \alpha_{Q_i} = ln - j$ where $l \geq 1$. Thus, it must be that $l - \alpha_P = -i$, i.e., $l + i = \alpha_P$. Let $v \in V(G)$ be chosen so that $\alpha_v = \max \{\alpha_w | w \in V(G) \setminus \{P\}\}$. Now, we consider two cases.

Case 1: Suppose $v = P_i$ for some $P_i \in U$.
Then, we have the following:

$$\begin{aligned}
\Delta_{P_i}(h) &= \sum_{i=1}^{n} \alpha_{Q_i} - \alpha_{P_i} n \\
&= lm - j - \alpha_{P_i} n \\
&= -j + (l - \alpha_{P_i})n.
\end{aligned}$$

We claim that $\alpha_{P_i} \geq l$. Suppose instead that $\alpha_{P_i} \leq l-1$. Then, $\sum_{i=1}^{n} \alpha_{Q_i} \leq \sum_{i=1}^{n}(l-1) = ln - n < ln - j$, which is a contradiction. Thus, the claim holds.

Hence, $\Delta_{P_i}(h) \leq -j$, which contradicts that $\Delta_{P_i}(h) \geq 0$.

Case 2: Suppose $v = Q_j$ for some $Q_j \in V$.

Consider $\Delta_{Q_j}(h) = \sum_{i=1}^{m-1} P_i + \alpha_P - \alpha_{Q_j} m$. We may assume that $\alpha_{Q_j} > \alpha_{P_i}$; otherwise we may apply Case 1. Hence, $\alpha_{P_i} - \alpha_{Q_j} \leq -1$. Also, note as in Case 1, it must be that $\alpha_{Q_j} \geq l$, and $l + i = \alpha_P$. Thus, $\alpha_P \leq \alpha_{Q_j} + i$.

Then,

$$\begin{aligned}
\Delta_{Q_j}(h) &= \sum_{i=1}^{m-1} P_i + \alpha_P - \alpha_{Q_j} m \\
&= \alpha_P - \alpha_{Q_j} + \sum_{i=1}^{m-1} (P_i - \alpha_{Q_j}) \\
&\leq \alpha_{Q_j} + i - \alpha_{Q_j} - (m-1) \\
&= i - (m-1) \\
&< 0.
\end{aligned}$$

Thus, it must be that such an α does not exist. Therefore, $H_f(P) = H_r(P) = \langle n, (m-1)n+1, (m-1)n+2, \ldots, (m-1)n+(n-1)\rangle$. □

We immediately have the following result.

Corollary 3.13. *Consider the complete bipartite graph $K_{m,m}$. Then, for any $P \in V(K_{m,m})$,*

$$H_f(P) = H_r(P) = \langle m, (m-1)m+1, (m-1)m+2, \ldots, m^2-1\rangle.$$

Based on the results above for families of graphs, one might be tempted to conjecture that $H_r(P) = H_f(P)$. As the next examples demonstrate, this is not the case in general.

Example 3.14. *Consider the graph G in Example 2.4. Note that $g = 1$. By Proposition 3.2, $|G_r(P)| = 1$ and if $\alpha \geq 2$, $\alpha \in H_r(P)$. Thus, $1 \in G_r(P)$ for all $P \in V(G)$. Now, consider the function f defined by $f(P_1) = -1$ and $f(P_i) = 0$ for $i = 2, 3, 4$. Then, $\Delta(f) = P_2 - P_1$. Hence, $1 \in H_f(P)$, which shows $H_r(P) \neq H_f(P)$.*

While it may seem that $1 \in H_f(P)$ can only occur if G has a leaf, the following proposition reveals that such problems may arise even if G does not contain a leaf.

Proposition 3.15. *Let $n \in \mathbb{Z}^+$. There exists a finite graph G so that $|H_f(P) \setminus H_r(P)| = n$ for some $P \in V(G)$.*

Proof. Let $n \in Z^+$. To construct such a graph let G_1 have genus g_1 and G_2 have genus g_2. Let G be the graph defined by constructing a bridge from a single vertex in each of G_1 and G_2 to a new vertex P. Then, G has genus $g_1 + g_2$ but $1 \in H_f(P_1)$. Thus, $|H_f(P) \setminus H_r(P)| = g_1 + g_2$. □

The following example shows a specific case of Proposition 3.15.

Example 3.16. *Consider the graph G shown in Figure 3.1.*

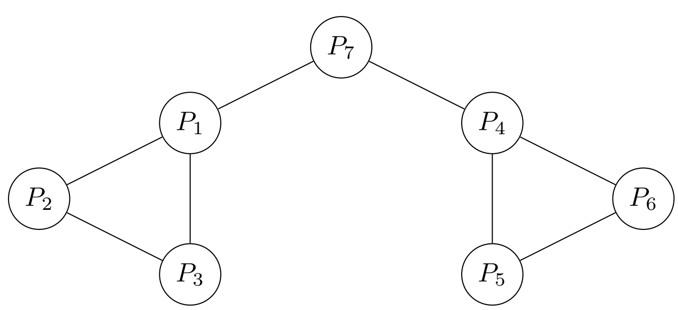

Figure 3.1: G.

Note that $g = 2$. Then, $2 = |G_r(P)|$ for all $P \in V(G)$ by Proposition 3.2. Now, consider $f \in \mathcal{M}(G)$ where $f(P_i) = 0$ for $i = 1, 2, 3, 7$ and $f(P_i) = 1$ for $i = 4, 5, 6$. Then, $\Delta(f) = P_5 - P_7$. Thus, $1 \in H_f(P_7)$. By Proposition 3.4, $H_f(P_7) = \mathbb{N}$.

The following example shows that even in the case of regular graphs, we do not always obtain $H_f(P) = H_r(P)$.

Example 3.17. *Consider the cube graph Q_3. Then, $\deg(P) = 3$ for all $P \in V(G)$. Thus, $3 \in H_f(P)$. However, if $Q \in V(G)$ so that Q is not a neighbor of P, one can verify that there is not a function $f \in \mathcal{M}(G)$ such that $\Delta(f) \geq Q - 3P$, which shows that $3 \notin H_r(P)$.*

4 Weierstrass semigroups of a multiple vertices on finite graphs

It is natural to ask if we may use similar tools to study semigroups arising from multiple vertices. In order to do so, we first consider the following definition of two analogues of $H_f(P)$ and $H_r(P)$. We observe that these analogues are similar to those in Definition 2.1.

Definition 4.1. *Let G be a graph and $P_1, P_2, \ldots, P_m \in V(G)$ be distinct vertices. Then,*

$$H_r(P_1, P_2, \ldots, P_m) = \left\{ \alpha \in \mathbb{N}^m \ : \begin{array}{l} r\left(\sum_{i=1}^m \alpha_i P_i\right) = r\left(\left(\alpha_j - 1\right) P_j + \sum_{i=1, i\neq j}^m \alpha_i P_i\right) + 1, \\ \text{for all } 1 \leq j \leq m \end{array} \right\}$$

and

$$H_f(P_1, P_2, \ldots, P_m) = \left\{ \alpha \in \mathbb{N}^m : \begin{array}{c} \text{there exists } f \in \mathcal{M}(G) \text{ such that} \\ \Delta(f) = A - \sum_{i=1}^n \alpha_i P_i \text{ where } A \geq 0 \text{ and } P_i \notin \mathrm{supp} A \text{ for all } i \end{array} \right\}.$$

An proof similar to that of Proposition 3.4 yields the following result.

Proposition 4.2. *Let G be a graph and $P_1, P_2, \ldots, P_m \in V(G)$. Then, $H_f(P_1, P_2, \ldots, P_m)$ is a numerical semigroup.*

Proof. Let $\alpha, \beta \in H_f(P_1, P_2, \ldots P_m)$. Then, there exist $f, g \in \mathcal{M}(G)$ such that $\Delta(f) = A_1 - \sum_{i=1}^m \alpha_i P_i$ and $\Delta(g) = A_2 - \sum_{i=1}^m \beta_i P_i$ where $A_1, A_2 \geq 0$, $P_i \notin \mathrm{supp} A_1$ and $P_i \notin \mathrm{supp} A_2$ for $1 \leq i \leq m$. Then, $\Delta(f + g) = A_1 + A_2 - \sum_{i=1}^m (\alpha_i + \beta_i) P_i$. Thus, $\alpha + \beta \in H_f(P_1, P_2, \ldots P_m)$. ☐

As before, we first consider the cycle on n vertices. We obtain the following result.

Theorem 4.3. *Let $n \geq 5$. Then if $P_1, P_2 \in V(C_5)$, $H_f(P_1, P_2) = H_r(P_1, P_2)$.*

In order to prove this, we need to first prove several lemmas.

Lemma 4.4. *Let $n \geq 5$. Then, if $P_1, P_2 \in V(C_n)$, $(1, 1) \in H_f(P_1, P_2)$.*

Proof. We have two possibilities to consider.

First suppose that P_1, P_2 are neighbors. Then, consider f_{P_1} and f_{P_2}. Then, $\Delta(f_{P_1} + f_{P_2}) = Q_1 + Q_2 - P_1 - P_2$ where $Q_1, Q_2 \in V(C_n) \setminus \{P_1, P_2\}$.

Next, suppose that P_1, P_2 are not neighbors. Then, there is a path A from P_1 to P_2 with at least 1 vertex distinct from P_1 and P_2. Denote the vertices other than P_1 and P_2 by P_i, $3 \leq i \leq k$. Define $f = \sum_{i=1}^k f_{P_i}$. Then, $\Delta(f) = Q_1 + Q_2 - P_1 - P_2$ where Q_1, Q_2 are two vertices, not necessarily distinct such that $Q_j \neq P_i$, $1 \leq i \leq k$. Hence, $(1, 1) \in H_f(P_1, P_2)$.

☐

Lemma 4.5. *Let $n \geq 5$. Then, if $P_1, P_2 \in V(C_n)$, $(1, 2) \in H_f(P_1, P_2)$ and $(2, 1) \in H_f(P_1, P_2)$.*

Proof. We show $(1,2) \in H_f(P_1,P_2)$ as a similar argument holds for $(2,1)$. We have two possibilities to consider. First suppose that P_1, P_2 are neighbors. Let Q_1 denote the other neighbor of P_1. Then, consider $f = 2f_{P_1} + 2f_{P_2} + f_{Q_1}$. Then, $\Delta(f) = A - P_1 - 2P_2$ where $A \geq 0$.

Next, suppose that P_1, P_2 are not neighbors. Then, since $n \geq 5$, there is a path A from P_1 to P_2 containing at least 2 other vertices. Let Q_1 be the vertex on A that is adjacent to P_2. Let $\{P_j\}_{j=3}^{l}$ denote the vertices that are not on A. Consider $f = f_{P_2} + \sum_{i=1}^{l} f_{P_i} + f_{Q_1}$. Then, $\Delta(f) = Q_k + P_3 + Q_2 - P_1 - 2P_2$, where Q_k a the neighbor of P_1, Q_2 is a neighbor of Q_1, and P_3 is a neighbor of P_2. By construction, Q_2 is distinct from P_1. Thus, $(1,2) \in H_f(P_1,P_2)$.

□

Lemma 4.6. *Let $n \geq 5$. Then, if $P_1, P_2 \in V(C_n)$, $(2,0),(3,0) \in H_f(P_1,P_2)$ and $(0,2),(0,3) \in H_f(P_1,P_2)$.*

Proof. We show $(2,0),(3,0) \in H_f(P_1,P_2)$ as a similar argument holds for $(0,2),(0,3) \in H_f(P_1,P_2)$.

Suppose P_1, P_2 are neighbors. Consider $f = 2f_{P_1} + f_{P_2}$. Then, $\Delta(f) = A - 3P_1$, where $A \geq 0$ and $P_1 \notin \mathrm{Supp}(A)$. Thus, $(3,0) \in H_f(P_1,P_2)$. Next, consider $g = 2f_{P_1} + f_{P_2} + f_{Q_1}$ where Q_1 is a neighbor of P_1. By assumption, Q_1 is not a neighbor of P_2. Hence, $\Delta(g) = A - 2P_1$, where $A \geq 0$ and $P_1 \notin \mathrm{Supp}(A)$.

Now, suppose Suppose P_1, P_2 are not neighbors. Then, there is a path from P_1 to P_2 containing at least two other vertices. Let Q_1 denote the vertex on this path that is a neighbor to P_1. Consider $f = 2f_{P_1} + f_{Q_1}$. Then,$\Delta(f) = A - 3P_1$, where $A \geq 0$ and $P_1, P_2 \notin \mathrm{Supp}(A)$. Moreover $\Delta(f_{P_1}) = A - 2P_1$, where $A \geq 0$ and $P_1, P_2 \notin \mathrm{Supp}(A)$. Hence, $(2,0) \in H_f(P_1,P_2)$. □

Using these results, we now prove Theorem 4.3. We note that Lemma 4.6 is necessary as $\alpha \in H_f(P_1)$ does not mean that $(\alpha, 0) \in H_f(P_1,P_2)$.

Proof. By the Riemann-Roch theorem for finite graphs, $H_r(P_1,P_2) = \{(\alpha,\beta) : \alpha = \beta = 0 \text{ or } \alpha+\beta \geq 2\}$. Moreover, if $(1,0),(0,1) \in H_f(P_1,P_2)$, $1 \in H_f(P_1)$ and $1 \in H_f(P_2)$, which we know is impossible. Thus, we must show that if $\alpha+\beta \geq 2$, $(\alpha,\beta) \in H_f(P_1,P_2)$. It suffices to show that $(0,2),(0,3),(1,2),(1,1),(2,1),(0,2),(0,3) \in H_f(P_1,P_2)$.

By Lemma 4.4, $(1,1) \in H_f(P_1,P_2)$.

□

Since we know that $H_r(P) \subseteq H_f(P)$, one might conjecture that in general,

$$H_r(P_1,P_2,\ldots,P_m) \subseteq H_f(P_1,P_2,\ldots,P_m).$$

However, the following example shows this fails in general. Moreover, we observe that if $n = 4$, this above result does not hold if P_1, P_2 are neighbors. Once can show that $(1,2) \notin H_f(P_1,P_2)$ by considering the image of the Laplacian.

We can also consider $H_f(P_1,P_2)$ for other families of graphs. Our main tool for this study will be the the Smith Normal Form of the Laplacian matrix of G. Using the Smith normal form of Q, we can solve the equation $Q\,[f] = [\Delta(f)]$, where $\Delta_{P_1}(f), \Delta_{P_2}(f) \leq 0$ and $\Delta_{P_i}(f) \geq 0$ for $i \neq 1,2$. Applying this approach to K_5, we obtain the following example.

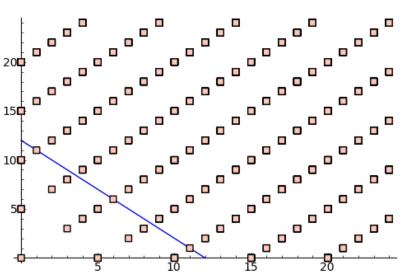

Figure 4.1: $H_f(P_1,P_2)$ for $P_1, P_2 \in V(K_5)$

Example 4.7. *Let $G = K_5$. The graph in Figure 4.1 shows ordered pairs (α, β) such that $(\alpha, \beta) \in H_f(P_1, P_2)$. The line occurs at $\alpha + \beta = 2g$. If $\alpha + \beta \geq 2g$, $(\alpha, \beta) \in H_r(P_1, P_2)$. If $H_r(P_1, P_2) \subseteq H_f(P_1, P_2)$, then all points to the right of the line would be in $H_f(P_1, P_2)$. However, $(15, 1) \notin H_f(P)$, for example. Hence, $H_f(P_1, P_2) \nsubseteq H_f(P_1, P_2)$.*

Thus, we can conclude less on the structure of these two sets in the case of multiple vertices. In fact, computation suggests that for any n, $H_f(P_1, P_2) \nsubseteq H_f(P_1, P_2)$ for K_n.

5 Conclusion

Inspired by the Riemann-Roch theorem for finite graphs, we explore two sets which provide analogues of the Weierstrass semigroup of a rational point on a nonsingular projective curve. In the case of a single vertex $H_r(P) \subseteq H_f(P)$. For a vertex P of the cycle C_n, the complete graph K_n, and the complete bipartite graph $K_{m,n}$, $H_r(P) = H_f(P)$. However, the difference $|H_f(P) \setminus H_r(P)|$ can be made arbitrarily large. While the notions of $H_r(P)$ and $H_f(P)$ can be extended to multiple vertices, less can be said about the relationship between these two sets. Moreover, it remains to show that $H_r(P_1, P_2, \ldots P_m)$ are numerical semigroups (even for $m = 1$).

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